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ACTION OF A ROTATING MAGNETIC FIELD ON A DIELECTRIC CYLINDER IMMERSED IN A MAGNETIC FLUID

A. F. Pshenichnikov and A. V. Lebedev

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1. Introduction. It is known [1–5] that in a magnetic fluid in a variable magnetic field tangential stresses occur which are due to the finite relaxation time of magnetization and its spatial nonuniformity. At the fluid boundary, magnetization undergoes a jump and the tangential stresses reach maximum values. Under these stresses the free boundary comes into motion, carrying along adjacent fluid layers and forming circulation hydrodynamic flow (rotational effect). Although this effect was observed experimentally as early as in 1967 [6], its quantitative description remains unsatisfactory.

The first attempts to calculate magnetic tangential flow at the boundary of an isothermic fluid were made by Tsebers et al. [1, 2] for Couette flow in the clearance between two coaxial cylinders and by Vislovich [7] for a plane layer. The calculation was performed in a uniform magnetic field approximation. This limits the range of applicability of the solution to weakly magnetic dilute solutions, and the presence of microscopic parameters related to individual colloidal particles allows one to estimate only the order of magnitude of the flow velocity. Lebedev et al. [3, 5] solved the problem using the equations of ferrohydrodynamics from [8–10], allowing for the spatial nonuniformity of magnetization and of the field and ignoring microscopic parameters. Such an approach allowed a more correct comparison of calculated and experimental data to be made, but the causes of the two- or threefold difference between the theory and experiment remain unclear.

The present work analyzes these causes of the differences, presents a more correct solution of the problem of coaxial cylinders, and gives new experimental data. The chosen geometry of the problem is convenient for both theoretical analysis and experimental studies. In particular, experiments with a free and elastically fixed inner cylinder allow one to determine the flow velocity and estimate the influence of a moving boundary on the hydrodynamics of a magnetic fluid.

2. Formulation of the Problem. Let a dielectric magnetic fluid fill the clearance between two vertical coaxial cylinders whose length is large in comparison with their diameters. The magnetization of both cylinders is thought to be negligible in comparison with the liquid magnetization. The outer magnetic field is uniform at a distance from the cylinders, is oriented perpendicular to their axis, and rotates in a horizontal plane with an angular velocity ω. The problem consists in finding the moment of the forces acting on the inner cylinder, and also the amplitude and profile of the hydrodynamic flow. The solution is based on the equations of ferrohydrodynamics [8–10], including an equation of motion that allows for nonequilibrium magnetization, the Maxwell equations for a magnetic field, and a relaxation equation for magnetization.

Here we restrict ourselves to the case of weak fields. This makes it possible to linearize the relaxation equation, to ignore the internal heat sources associated with energy dissipation of the rotating field, and to consider the flow slow and stationary. The system of ferrohydrodynamic equations subject to these conditions can be written as

$$-\nabla p + \eta\Delta v + \mu_0\left[(\mathbf{M}\nabla)\mathbf{H} + \frac{1}{2}\,\mathrm{rot}\,(\mathbf{M}\times\mathbf{H})\right] = 0; \tag{2.1}$$

$$\mathrm{rot}\,\mathbf{H} = 0, \qquad \mathrm{div}\,\mathbf{B} = 0, \qquad \mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}); \tag{2.2}$$

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$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{1}{\tau}(\mathbf{M} - \chi_0 \mathbf{H}); \tag{2.3}$$

$$\operatorname{div} \mathbf{v} = 0. \tag{2.4}$$

Here $\mathbf{v}$ is the flow velocity; $\mathbf{M}$ is the fluid magnetization; $\mathbf{H}$ and $\mathbf{B}$ are the intensity and induction of the field; $\mu_0 = 4\pi \cdot 10^{-7}$ H/m; χ_0 is the equilibrium susceptibility; and τ is the relaxation time of the magnetic moment of a particle. Relaxation equation (2.3) takes into account the fact that the flow vorticity is small in comparison with the field-rotation frequency. The term with $\operatorname{rot} \mathbf{v}$ is therefore absent from the right-hand side of Eq. (2.3). The low-vorticity condition is well satisfied in practice, which allows the magnetic part of the problem to be solved independently from the hydrodynamic part.

The boundary conditions for the velocity are conventional. In the cylindrical coordinates r and φ, they take the form

$$v = \Omega R_1 \quad \text{at} \quad r = R_1, \quad v = 0 \quad \text{at} \quad r = R_2 \tag{2.5}$$

(R_1 and R_2 are the radii of the inner and outer cylinders, respectively, and Ω is the angular rotation velocity of the inner cylinder). The outer cylinder is immobile. According to [8], the tangential stresses in the magnetic fluid are determined by the tensor

$$\sigma_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) + H_i B_k + \frac{\mu_0}{2}(M_i H_k - M_k H_i).$$

This expression and the fact that the normal induction component and the tangential component of field intensity are continuous at the fluid boundary allow one to write the boundary condition for tangential stresses (at $r = R_1$ and R_2) in the form

$$\sigma'_{ik} = \frac{\mu_0}{2}(M_i H_k - M_k H_i) + \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right), \tag{2.6}$$

where σ'_{ik} is the mechanical stress in the cylinder wall. In particular, if the inner cylinder is free to rotate ($\sigma'_{ik} = 0$), condition (2.6) determines the amplitude of hydrodynamic flow.

3. Solution of the Problem and Analysis of the Results. First, note an important feature of the weak-field approximation: the volume magnetic forces in a homogeneous fluid have a gradient form [3, 9]. Indeed, using the Maxwell equation (2.2) and the vector identities in [11], we find

$$\mathbf{F} = \mu_0 \left[(\mathbf{M}\nabla)\mathbf{H} + \frac{1}{2} \operatorname{rot} (\mathbf{M} \times \mathbf{H}) \right] = \frac{\mu_0}{2} \left[\nabla(\mathbf{M} \cdot \mathbf{H}) - \mathbf{H} \times \operatorname{rot} \mathbf{M} + (\mathbf{H} + \mathbf{M}) \operatorname{div} \mathbf{H} \right].$$

On the other hand, it follows from (2.2) and (2.3) with uniform χ_0 and τ (the fluid is isothermic) that $\operatorname{div} \mathbf{H} = 0$ and $\operatorname{rot} \mathbf{M} = 0$. Thus, the volume magnetic forces lead only to pressure renormalization. Only the tangential magnetic stresses on the mobile boundary can be a source of stationary motion.

Bearing the above remark in mind, we find a simple Couette flow profile from Eqs. (2.1) and (2.4) and no-slip condition (2.5):

$$v(r) = \frac{\Omega R_1^2 (R_2^2 - r^2)}{r(R_2^2 - R_1^2)}, \tag{3.1}$$

where $v(r)$ is the azimuthal velocity component; Ω is determined from condition (2.6). The problem is thus reduced to calculating the fields $\mathbf{H}$ and $\mathbf{M}$ inside the fluid.

In the case of long coaxial cylinders, Maxwell equations (2.2) and relaxation equation (2.3) admit a simple solution. In a coordinate system that rotates with the field, it takes the form

$$\begin{gathered} H_r = C_1 \cos\varphi - \frac{C_2}{r^2} \cos(\varphi + \beta), \quad H_\varphi = -C_1 \sin\varphi - \frac{C_2}{r^2} \sin(\varphi + \beta), \\ M_r = \chi \left[C_1 \cos(\varphi + \delta) - \frac{C_2}{r^2} \cos(\varphi + \beta + \delta) \right], \quad M_\varphi = -\chi \left[C_1 \sin(\varphi + \delta) + \frac{C_2}{r^2} \sin(\varphi + \beta + \delta) \right]. \end{gathered} \tag{3.2}$$

Here $\tan \delta = \chi_2/\chi_1$; χ is the dynamic susceptibility modulus. The real part χ_1 and the imaginary part χ_2 of the dynamic susceptibility are described by the Debye formulas

$$\chi_1 = \chi_0/(1 + \omega^2\tau^2), \qquad \chi_2 = \chi_0 \omega\tau/(1 + \omega^2\tau^2).$$

The constants β, C_1, and C_2 are found from the boundary conditions for the intensity **H** and induction **B**. Skipping, for the sake of brevity, awkward intermediate calculations, we write the result:

$$\frac{C_1}{\sqrt{4+4\chi_1+\chi^2}} = \frac{C_2}{\chi R_1^2} = \frac{2H_0}{4+4\chi_1+\chi^2-\chi^2 R_1^2/R_2^2}, \quad \tan\beta = 2\chi_2/(\chi^2+2\chi_1). \tag{3.3}$$

The solution for the outer region is similar in structure to (3.2), but with different constants and phases. The field inside the inner cylinder is uniform.

It is pertinent to note that actual ferrocolloids are essentially polydispersed, and that each particle fraction is characterized by its own relaxation time of the magnetic moment. The time spectrum is usually very wide (several orders [12]). For this reason, it is necessary to write an individual relaxation equation for each fraction of particles. But if we do this and add up the contributions of different fractions to magnetization, we will again come to Eqs. (3.2), with the only difference that χ_1 and χ_2 are no longer described by the Debye equations and a relaxation time distribution function should be defined to calculate them. This, however, involves no difficulties, since the procedure of measuring dynamic susceptibility is well developed and there is no need for calculation of χ_1 and χ_2.

Substitution of (3.1)–(3.3) into (2.6) at $\sigma'_{ik} = 0$ yields

$$\Omega = K(1 - R_1^2/R_2^2)/4\eta; \tag{3.4}$$

$$K = -\frac{16\mu_0\chi_2(1+\chi_1)H_0^2}{(4+4\chi_1+\chi^2-\chi^2R_1^2/R_2^2)^2}. \tag{3.5}$$

Here K is the specific (per volume unit) moment of the magnetic forces acting on the inner cylinder. The minus sign on the right-hand side of (3.5) means that this moment is opposite in direction to the rotation of the outer field. The inner cylinder is also counterrotating with respect to the field.

The interactions among colloidal particles in dilute solutions with a low concentration of the magnetic phase are insignificant and the equilibrium susceptibility χ_0 grows linearly with increasing particle concentration n. For concentrated solutions, the dependence $\chi_0(n)$ is nearly parabolic [12]. The rapid growth in susceptibility with an increase in particle concentration leads to a nonmonotone dependence $K(n)$ [formula (3.5)]. For low concentrations, the torque increases on account of growth in χ_2, and for high concentrations it decreases due to a rapid growth in the denominator. The major role here is played by the demagnetizing field of the sample. The results in [1, 2, 9], which were obtained for the case where demagnetizing fields were ignored, coincide with our results in the limit $\chi \to 0$. The demagnetizing field of the sample is taken into account in [3, 5], but the results for K differ from (3.5) in terms that are quadratic in χ. In the case of dilute solutions ($\chi < 1$), the corresponding additional terms are not large, but at a high concentration of the magnetic phase they can lead to a two- to fivefold difference in the results. And this is indeed the case.

The difference between the results of the present work and those of [3, 5] is, in our opinion, caused mainly by the initial equations and assumptions. In solving the problem in [3, 5], the relaxation equation was dropped and replaced by the assumption of constancy of the angle between the vectors **H** and **M**. This assumption agrees well with Eq. (2.3) when the magnetic field is polarized circularly, that is, when the samples have the shape of a solid cylinder or a sphere. However, the magnetic fluid in the clearance between coaxial cylinders is characterized by an elliptic polarization of the magnetic field. In this case, a rigid-bond approximation for **H** and **M** and relaxation equation (1.3) yield different results.

4. Experimental Results and Their Comparison with Theory. Experiments were performed with glass cylinders according to the procedure in [3]. The outer cylinder was 33 mm in diameter and 200 mm high. The inner cylinder had the shape of an areometer. It was suspended in the fluid so that only a thin capillary with marks on it extended above the surface. The cylinder diameter was 4.5 mm and the height h = 60 mm. In measurements of the moment of magnetic forces, the inner cylinder was suspended on a thin elastic strip of beryllium bronze. Colloidal magnetite solutions in liquid hydrocarbons were used as the working fluid. They were prepared under laboratory conditions using a standard chemical precipitation method. The viscosity of the solutions was measured with a rotation viscosimeter of the "Reotest-2" type. Dynamic susceptibility was measured by a mutual-inductance bridge in a weak (200 A/m) plane-polarized

TABLE 1

H_0, kA/m	K_0, N/m^2 (experiment)	K_0, N/m^2 (calculation)
0.31	0.0044	0.0041
0.61	0.017	0.016
0.92	0.040	0.037
1.23	0.071	0.065
1.54	0.108	0.102
1.84	0.155	0.147
2.15	0.211	0.200
2.46	0.278	0.261
2.76	0.347	0.330
3.07	0.436	0.407
3.84	0.674	0.636
4.61	0.960	0.916

field in accordance with the procedure of [13] at the frequencies that were used in the experiments with a rotating field. Crossed Helmholtz coils were used to create the rotating field. The rotating field intensity in different experiments was varied from 0 to 5 kA/m, and the rotation frequency, from 120 to 1,000 Hz.

To make an additional check of the installation quality and to estimate the measurement error, control experiments were performed to measure the moment of forces acting on a magnetic fluid sample in the form of a solid cylinder. In these experiment, a round test tube with diameter $d = 3.9$ mm and height $h = 61$ mm was filled with a magnetic fluid. The moment of forces acting on the immobile cylinder can be expressed from (2.6), (3.2), and (3.3) for $R_1 = 0$ and $v = 0$:

$$K_0 = \frac{4\mu_0\chi_2 H_0^2}{4 + 4\chi_1 + \chi^2} = \mu_0\chi_2 H_1^2 \tag{4.1}$$

where H_1 is the field intensity inside the fluid. Due to the demagnetizing factor, the magnitude of H_1 was, as a rule, several times smaller than that of the outer field H_0.

Typical results of the control experiments are presented in Table 1. The experiments were performed at a rotation frequency of 400 Hz for $\chi_1 = 5.13$ and $\chi_2 = 0.437$ (a highly concentrated fluid) and various field magnitudes. As could be expected, the moment of the magnetic forces grows with field intensity by a quadratic law. However, the value of K_0 calculated from (4.1) is 5–8% smaller than the experimental value. This difference is mainly due to the edge effects, which are ignored in the theory. Obviously, their relative contribution is of the order of d/h and can be determined more precisely in experiments with samples of different lengths.

The results of the control experiment that were obtained for $H_1 = 860$ A/m in the present work are presented in Fig. 1 (the dots). As is seen from Fig. 1, the torque grows with the d/h ratio by a nearly linear law (the line). The value of K_0 that corresponds to the limit $h \to \infty$ is obtained by extrapolation of the averaged line to the coordinate axis and is 9% smaller than the value corresponding to $h = 61$ mm. If we now introduce a corresponding correction into the experimental data in Table 1, the values of K_0 will be, on the average, only 3% smaller than the calculated values. This difference should be deemed insignificant, inasmuch as the measurement error of χ_2 from formula (4.1) is also within 3–5%.

The results of experiments with immobile coaxial cylinders and their comparison with the calculation are presented in Fig. 2, wherein the curve corresponds to formula (3.5) and the dots to the experimental data. The experimentally measured moments of forces acting on the inner cylinder were, on the average, 8% larger than the calculated values, but after introducing corrections for the edge effects this difference decreased to 3–4%. Similar results were obtained at rotation frequencies of the outer field of 120, 200, and 1,000 Hz. Thus, the situation is entirely similar analogous to that observed in experiments with a sample in the form of a solid cylinder. As for the formula for the torque obtained in [3, 5] for a circularly polarized field, it yields results

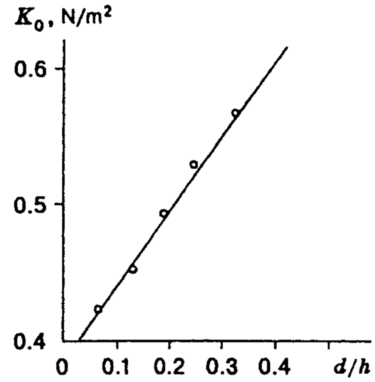

Fig. 1

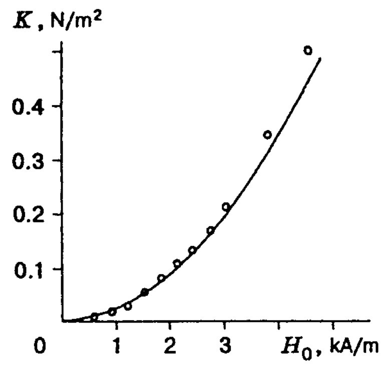

Fig. 2

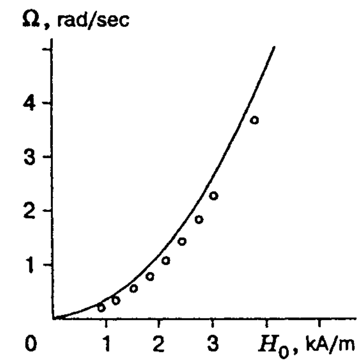

Fig. 3

that are overstated by a factor of three. The agreement of this formula with experimental data, as noted in one experiment in [3], turned out to be accidental and was due to a procedural error in [3] in data processing.

The angular rotation velocity of the inner cylinder with the immobile outer cylinder is presented in Fig. 3 for a sample with an effective viscosity $\eta = 0.0172$ Pa·sec and different intensities of the outer field. Here the curve corresponds to formula (3.4), whose accuracy proved to be far lower than the accuracy of formula (3.5) for the torque (the experimental data are shown by the dots). The calculated value of the cylinder's rotation velocity is 15–20% higher than the experimental one. This difference is, in our opinion, most likely caused by the fact that the concentrated magnetic fluids were weakly non-Newtonian. Indeed, their effective viscosity in the initial portion of the rheological curve can exceed significantly the viscosity measured by a standard viscosimeter [14]. It is precisely the flow with low shear velocities (not greater than 10 sec^{-1}) that was realized in experiments with coaxial cylinders. This is two orders smaller than in the viscosimetric measurements. In addition, the relative deviation of the calculated value of Ω from the experimental value decreases monotonically in our experiments with an increased velocity of the inner cylinder. This also can be regarded as a consequence of a decrease in the effective viscosity with an increased shear velocity.

The amplitude of the outer field H_0 did not exceed 4–5 kA/m in our experiments. This limitation was due to the fact that the relaxation time in Eq. (2.3) could be considered constant only in weak fields: the Langevin parameter ($\xi = \mu_0 mH/kT$, m is the magnetic moment of a single-domain colloidal particle, k is the Boltzmann constant, and T is the absolute temperature) must be small compared with unity [8]. For $\xi \geqslant 1$, linear Eq. (2.3) must be replaced by a more general one. For magnetite fluids, the magnetic moment m is on the order of 10^{-19} A·m² and the condition $\xi < 0.1$ yields an estimate for the maximum field intensity inside the fluid and for the limits of applicability of formulas (3.4) and (3.5): $H_1 < 1$ kA/m. Although the outer field H_0 can be higher by a factor of two to four, it nevertheless must not exceed a few kiloampers per meter. As is seen from Table 1 and Figs. 2 and 3, most of our experiments were performed under conditions that could be called the limiting allowable conditions within the assumptions made. With a further increase in the field, a comparison of the experimental and theoretical results would be improper.

Thus, the results obtained in this work show that ferrohydrodynamic equations (2.1)-(2.4) with boundary condition (2.6) can be a good basis for calculating magnetic fluid flow in variable magnetic fields and for describing adequately the behavior of actual ferrocolloids. In the case of weak fields, one excludes from consideration the difficult-to-determine microscopic parameters related to single colloidal particles (the Langevin parameter and the magnetic moment relaxation time). The torque acting on a dielectric cylinder immersed in a magnetic fluid and the flow amplitude are expressed in terms of the imaginary and real parts of the dynamic susceptibility and agree satisfactorily with experimental values. A more precise comparison of the experimental and theoretical results requires much greater effort to develop a new procedure for measuring magnetic susceptibility and recording the rheological curve at limiting low shear velocities. In the case of a magnetic field of arbitrary intensity, it is necessary to consider a nonlinear relaxation equation and take into account the possible nonpotentiality of volume magnetic forces, a wide spectrum of relaxation times, and their dependence on the field. An analytical solution of the problem in such a general formulation seems to be impossible.

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REFERENCES

1. A. O. Tsebers, "Interface stresses in the hydrodynamics of fluids with internal rotations," in: *Magnetohydrodynamics*, No. 1, 79–82 (1975).
2. A. O. Tsebers, "Moment stresses and hydrodynamics of a magnetic fluid in uniform rotating magnetic fluids," in: *Magnetohydrodynamics*, No. 4, 9–13 (1978).
3. A. V. Lebedev and A. F. Pshenichnikov, "On magnetic fluid motion in a rotating field," in: *Magnetohydrodynamics*, No. 1, 7–12 (1991).
4. R. E. Rosensweig, J. Popplewell, and R. J. Johnston, "Magnetic fluid motion in a rotating field," *J. Magn. Magn. Mater.*, **85**, 171–180 (1990).
5. A. V. Lebedev and A. F. Pshenichnikov, "Rotational effect: The influence of free or solid moving boundaries," *J. Magn. Magn. Mater.*, **122**, 227–230 (1993).
6. R. Moskowitz and R. E. Rosensweig, "Nonmechanical torque-driven flow of a ferromagnetic fluid by an electromagnetic field," *Appl. Phys. Lett.*, **11**, No. 10, 301–306 (1967).
7. A. N. Vislovich, "Action of a rotating magnetic field on a ferromagnetic suspension in a layer with a free boundary," *Pis'ma Zh. Tekh. Fiz.*, **1**, No. 16, 744–748 (1975).
8. M. I. Shliomis, "Magnetic fluids," *Usp. Fiz. Nauk*, **112**, No. 3, 427–458 (1974).
9. V. G. Bashtovoi, B. M. Berkovskii, and A. N. Vislovich, *Introduction to Magnetic Fluid Thermomechanics*, IVTAN, Moscow (1985).
10. M. I. Shliomis, T. P. Lyubimova, and D. V. Lyubimov, "Ferrohydrodynamics: an essay on the progress of ideas," *Chem. Eng. Comm.*, **67**, 275–290 (1988).
11. G. Arfken, *Mathematical Methods in Physics* [in Russian], Atomizdat, Moscow (1970).
12. M. I. Shliomis, A. F. Pshenichnikov, K. I. Morozov, and I. Yu. Shurubor, "Magnetic properties of ferrocolloids," *J. Magn. Magn. Mater.*, **85**, 40–46 (1990).
13. A. F. Pshenichnikov, "Dispersion of magnetic susceptibility of magnetic colloids," in: *Nonsteady Processes in Magnetic Suspensions* [in Russian], Sverdlovsk (1986).
14. V. G. Gilev, "Experimental study of the rheological properties of magnetic fluids," Candidate's thesis [in Russian], Perm' State Univ., Perm' (1987).